2.40. Derived Rules of Inference

While the deductive system developed so far reliably provides a deduction for all (and only) the valid arguments in the formal language, many of these deductions are quite long. Here we develop a method of deductive ‘shortcuts’ to reduce the size and complexity of deductions, by supplementing the deductive system with convenient further rules.

1. Derived Rules. We established the validity of the following little argument by constructing an indirect deduction of it.

1. $\sim P$

$\therefore\ \sim(P \wedge Q)$

1.	P	
		Get $\sim(P \wedge Q)$ (ID)
2.	$\sim\sim(P \wedge Q)$	AID
3.	$(P \wedge Q)$	2, $\sim-$
4.	P	3, $\wedge-$
5.	$\sim P$	1, R
6.	$\sim(P \wedge Q)$	2, 4, 5, ID

Now this general argument pattern reappears in larger arguments such as the following.

1. $(R \wedge \sim P)$

$\therefore\ (R \wedge \sim(P \wedge Q))$

We can certainly build a deduction of this argument in our current deductive system. But doing so would just involve disassembling the conjunction on Line 1 using $\wedge-$; deducing “$\sim(P \wedge Q)$” from “$\sim P$” in an ID; then assembling the conclusion “$(R \wedge \sim(P \wedge Q))$” from its two parts via $\wedge+$.

We’re really just pasting our earlier deduction of “$\sim P \therefore \sim(P \wedge Q)$” into the middle of this larger deduction.

1.	$(R \wedge \sim P)$	
	Get: $(R \wedge \sim(P \wedge Q))$	
2.	R	1, $\wedge-$
3.	$\sim P$	1, $\wedge-$
	Get: $\sim(P \wedge Q)$ (ID)	
4.	$\sim\sim(P \wedge Q)$	AID
5.	$(P \wedge Q)$	4, $\sim-$
6.	P	5, $\wedge-$
7.	$\sim P$	3, R
8.	$\sim(P \wedge Q)$	4, 6, 7, ID
9.	$(R \wedge \sim(P \wedge Q))$	2, 8, $\wedge+$

Yet we could bypass this repeated labor by instead treating the valid argument “$\sim P \therefore \sim(P \wedge Q)$” as immediately justified by an additional **rule of inference**. That would mean adding the following general argument pattern to our stock of inference rules.

Negation-Conjunction ($\sim\wedge$)

$$\frac{\sim \bullet}{\sim(\bullet \wedge \blacktriangle)}$$

Then the deduction is simplified like so.

1.	$(R \land \sim P)$	
		Get: $(R \land \sim(P \land Q))$
2.	R	1, $\land -$
3.	$\sim P$	1, $\land -$
4.	$\sim(P \land Q)$	3, $\sim\land$
5.	$(R \land \sim(P \land Q))$	2, 4, $\land +$

We're confident that using this new rule wouldn't compromise the validity of the step in the deduction, since we could always instead paste in the deduction of its conclusion from its premise, as we did in our original deduction of "$(R \land \sim(P \land Q))$" from "$(R \land \sim P)$". Such a rule – added to the system, and justified by a deduction – is a **derived rule** (in contrast with the **basic rules** of the deductive system, which are 'basic' precisely because they're not justified by a deduction)[1]. Since the system is capable of all the same deductions without them, derived rules aren't essential parts of the deductive system – just convenient shortcuts.

Suppose we call our original system of deduction (without Rule $\sim\land$) **System 2** ("2" for the chapter in which it's presented). And let **System 2.1** be the system just like System 2 but with $\sim\land$ as an additional **basic** (non-derived) rules. Whenever System 2.1 invokes its rule $\sim\land$, System 2 can paste in its deduction matching that rule. So the same set of arguments are recognized as valid by both systems – the only difference being that the System 2 deduction will be longer when pasting in the $\sim\land$ deduction.

We say that two systems are **deductively equivalent** when they pick out exactly the same arguments as valid. Being deductively equivalent to System 2, System 2.1 has no advantage except convenience: where an inference of the $\sim\land$ form is involved, the System 2.1 deduction will be shorter. On the other hand, from the

[1] Of course, in justifying the derived rule by a deduction, that deduction must not appeal to the very rule being justified, or the justification will be circular. So in the above deduction of "$\sim(P \land Q)$" from "$\sim P$," we were careful not to appeal to the rule $\sim\land$ anywhere in the deduction.

perspective of System 2 the stock of inference rules in System 2.1 contain excess baggage – since throwing the rule **~∧** overboard brings no loss of deductive power.

A slightly leaner third system – call it “**System 2.2**” – offers another illustration of deductive equivalence. System 2.2 is like our System 2, except that it **has the ~∧ rule and lacks our ∧– rule**. It might seem that certain valid arguments would escape the grasp of System 2.2 deductions– most obviously arguments of the following sort.

1. (P ∧ Q)

∴ P

But that is not so. For System 2.2 has a deduction of this argument using only its basic inference rules.

1.	(P ∧ Q)	
		Get: P (ID)
2.	~P	AID
3.	~(P ∧ Q)	2, ~∧
4.	(P ∧ Q)	1, R
5.	P	2, 3, 4, ID

Systems 2 and 2.2 are thus **deductively equivalent**. (A user of System 2.2 could, if she wished, treat our rule ∧– as a derived rule.) The existence of different, yet equivalent, deductive systems shows that we have some latitude concerning which deductive system we use to pick out the valid arguments. In this respect choice of deductive system is similar to our earlier choice among expressively equivalent formal languages.[2] In both cases, systems with quite different basic elements nonetheless prove equivalent.

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[2] In 2.30; and later in 3.9 through 3.12.

2. De Morgan's Law. Our point in discussing the rule $\sim\land$ was only to illustrate the concept of a derived rule. We won't bother adding $\sim\land$ to our deductive system – making the judgment call that the convenience it brings is insufficient to justify cluttering up our list of inference rules.

But a different inference rule is useful enough to merit adding as a derived rule: **De Morgan's Law**.

De Morgan's Law (DM)

Inward DM **Outward DM**

$$\frac{\sim(\bullet \lor \blacktriangle)}{(\sim\bullet \land \sim\blacktriangle)} \qquad \frac{\sim(\bullet \land \blacktriangle)}{(\sim\bullet \lor \sim\blacktriangle)} \qquad\qquad \frac{(\sim\bullet \land \sim\blacktriangle)}{\sim(\bullet \lor \blacktriangle)} \qquad \frac{(\sim\bullet \lor \sim\blacktriangle)}{\sim(\bullet \land \blacktriangle)}$$

We encountered these four valid argument patterns earlier as semantic equivalences.[3] But now they act as two types of inference rule: **inward DM**, which pushes a tilde into the parts of a disjunction or conjunction; and **outward DM**, which 'extracts' a tilde from the parts of a disjunction or conjunction. (We bother to label the two varieties of DM because they play different roles in deduction.)

Most obviously: inward DM allows for the easy dispatch of otherwise vexing AIDs like the following.

1. $(\sim P \lor R)$
2. $(\sim Q \lor S)$
3. $(P \lor Q)$

Get: $(R \lor S)$ (ID)

4. | $\sim(R \lor S)$ AID

[3] In 2.17 § 1.

Armed only with the seven deductive rules and ID, the situation looks bleak. The only move open to us here is to start a second ID within the first.

But with DeMorgan’s Law that fearsome AID is immediately tamed.

1.	$(\sim P \vee R)$	
2.	$(\sim Q \vee S)$	
3.	$(P \vee Q)$	
		Get: $(R \vee S)$ (ID)
4.	$\sim(R \vee S)$	AID
5.	$(\sim R \wedge \sim S)$	4, In DM

What then follows is a thoroughly automatic cascade of Elim rules, backing its way into contradictory sentences.

1.	$(\sim P \vee R)$	
2.	$(\sim Q \vee S)$	
3.	$(P \vee Q)$	
		Get: $(R \vee S)$ (ID)
4.	$\sim(R \vee S)$	AID
5.	$(\sim R \wedge \sim S)$	4, In DM
6.	$\sim R$	5, $\wedge-$
7.	$\sim S$	5, $\wedge-$
8.	$\sim Q$	2, 7, $\vee-$
9.	P	3, 8, $\vee-$
10.	$\sim P$	1, 6, $\vee-$
11.	$(R \vee S)$	4, 8, 9, ID

Of course, to use De Morgan’s Law as a legitimate derived rule we first need to build deductions establishing that the conclusion is indeed deducible from the premise in each case. Here’s the deduction of the form of DM just used.[4]

1.	~(P ∨ Q)	
	Get: (~P ∧ ~Q)	(ID)
2.	~(~P ∧ ~Q)	AID
	Get: ~P	(ID)
3.	~~P	AID
4.	P	3, ~−
5.	(P ∨ Q)	4, ∨+
6.	~(P ∨ Q)	1, R
7.	~P	3, 5, 6, ID
	Get: ~Q	(ID)
8.	~~Q	AID
9.	Q	8, ~−
10.	(P ∨ Q)	9, ∨+
11.	~(P ∨ Q)	1, R
12.	~Q	8, 10, 11, ID
13.	(~P ∧ ~Q)	7, 12, ∧+
14.	~(P ∧ Q)	2, 13, ID

With DM added to our deductive system we’re in a position to simplify the negation of any molecular sentence.[5]

[4] Note that since Line 2 isn’t cited in the justification of Lines 3 through 13, we deduced Line 13 without using Line 2. We could thus have avoided using ID to deduce “(~P ∧ ~Q),” instead proceeding directly to the smaller IDs for “~P” and “~Q”. In that case Line 13 would be the last line of the deduction, and the large ID box (with its AID Line 2) wouldn’t appear – shaving two lines off the deduction.

[5] The negation of a negation is already handled by the rule ~−.

3. Deductive Strategy, Revisited. In terms of strategy, Inward DM is of particular use in making an AID manageable. Whenever an ID begins with a negated conjunction or negated disjunction, we now automatically apply inward DM to yield a disjunction or conjunction susceptible to Elim rules. For instance, the AID “~(P ∧ Q)” becomes “(~P ∨ ~Q)” (and ∨– is then applied, if possible); while the AID “~(P ∨ Q)” becomes “(~P ∧ ~Q)” (with ∧– then applied).

Of course Inward DM proves handy for sentences other than an AID. In general, **Inward DM is an Elim rule** – in the sense that it can’t be applied an unlimited number of times, and so can safely be executed whenever possible.

Outward DM will also have only a finite number of applications in a given deduction; so can Outward DM can likewise be trusted not to run amok. Still, our strategy will be to use outward DM primarily as a ‘setup’ rule, in the same spots, and for the same reasons, as the Intro rules. For Outward DM yields us a sentence – a negated conjunction or negated disjunction – to which an Elim rule won’t automatically apply. So we use Outward DM chiefly to get a missing sentence needed when the deduction has ground to a halt: to complete an instance of ∨–; or to get half of a contradiction when using ID or the sentence on the “Get” line otherwise.

The addition of DM to the deductive system streamlines deductions so much that adding further derived rules won’t prove necessary. Our system of Chapter 2 deduction has thus reached its final form.

Summary: DeMorgan’s Law Strategy

Inward DM (In DM)

$$\frac{\sim(\bullet \vee \blacktriangle)}{(\sim\bullet \wedge \sim\blacktriangle)} \qquad \frac{\sim(\bullet \wedge \blacktriangle)}{(\sim\bullet \vee \sim\blacktriangle)}$$

Outward DM (Out DM)

$$\frac{(\sim\bullet \wedge \sim\blacktriangle)}{\sim(\bullet \vee \blacktriangle)} \qquad \frac{(\sim\bullet \vee \sim\blacktriangle)}{\sim(\bullet \wedge \blacktriangle)}$$

- Treat **Inward DM** as an **Elim rule**: use whenever possible. In particular: automatically apply inward DM when the AID is a negated conjunction or negated disjunction.

- Treat **Outward DM** as an **Intro rule**: to supply (i) the missing part to apply an Elim rule; (ii) the sentence on the “Get” line; or half of a contradiction (when using an ID).